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Weakly nonlinear waves in a stratified fluid: a variational formulation

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The Lagrangian L for gravity waves of small but finite amplitude in an N-layer stratified fluid is constructed as a function of the generalized coordinates $\boldsymbol{q}_\nu(t) \equiv \{q_n^\nu(t)\}$ of the $N+1$ interfaces, where the q_n^ν are the Fourier coefficients of the expansion of the interfacial displacement $\eta_\nu(\boldsymbol{x}, t)$ in a complete, orthogonal set $\{\psi_n(\boldsymbol{x})\}$. The density is constant in each layer, by virtue of which a velocity potential exists for that layer (even though the full flow is rotational). The explicit expansion of L is constructed through fourth-order in $\boldsymbol{q}_\nu$ and $\dot{\boldsymbol{q}}_\nu$ through an extension of the surface-wave formulation (Miles 1976), in which the pressure appears as the Lagrangian density (Luke 1967). Three-dimensional progressive and standing interfacial waves in a two-layer fluid are treated as general examples, and the two-dimensional results of Hunt (1961) and Thorpe (1968) are recovered as explicit examples. It is shown that the spatial resonance between surface and internal waves conjectured by Mahony & Smith (1972) is impossible for the two-layer Boussinesq model.

The joint limit $N\uparrow\infty$ and layer thickness$\downarrow 0$ yields the Lagrangian density L for a continuously stratified, Boussinesq fluid as a functional of $q_n(y)$ and $\dot{q}_n(y)$, where y, the counterpart of the layer index, is a Lagrangian (rather than Eulerian) coordinate. The coefficient C in the nonlinear dispersion relation $(\omega/\omega_1)^2 = 1+Ck^2A^2$ for progressive waves of frequency ω, wavenumber k and amplitude A, where $\omega_1 = \omega_1(k)$ for infinitesimal waves, is determined for any density profile for which the (linear) vertical structure problem can be solved. Explicit results are given for a fluid of finite vertical extent in which the buoyancy frequency is constant and for a vertically unbounded fluid in which the buoyancy frequency varies like sech (y/h) and $C = C(kh)$.

1. Introduction

The primary purpose of the following development is the construction of the Lagrangian for gravity waves of small but finite amplitude in a stratified, inviscid fluid with a rigid bottom, a free upper surface and a cylindrical boundary (which includes the limiting case of a laterally unbounded fluid) as an explicit function of an appropriate set of generalized coordinates. [Those earlier Hamiltonian formulations for stratified flow with which I am familiar – e.g. Seliger & Whitham (1968), Milder (1982) and Henyey (1983) – are concerned primarily with the general form of the Hamiltonian or Lagrangian functionals from which the equations of motion can be derived through Hamilton's principle, rather than with explicit representations in generalized coordinates.] The extension to stratified shear flows is considered in the following paper (Miles 1986).

Weakly nonlinear, two-dimensional waves in a stratified fluid have been considered by Hunt (1961) and Thorpe (1968*a*, *b*). Hunt obtained third-order (in amplitude) approximations to the profiles and quadratic approximations to the frequencies of interfacial progressive waves in a two-layer fluid with rigid upper and lower boundaries and the corresponding results for standing waves in a vertically unbounded fluid. Thorpe (1968*a*) extended Hunt's results for standing waves to a two-layer fluid of finite vertical extent and obtained second-order descriptions of both standing (1968*a*) and progressive (1968*b*) waves in a continuously stratified fluid using the Boussinesq approximation; however, he did not establish the amplitude dependence of dispersion for progressive waves. Both Hunt and Thorpe used perturbation expansions of the type pioneered by Stokes and Levi-Civita.

Tsuji & Nagata (1973) have extended Hunt's expansion to fifth order for progressive waves on the interface between two vertically unbounded fluids. Holyer (1979) has determined the wave of maximum amplitude, and Meiron & Saffman (1983) have obtained numerical solutions for overhanging waves, for this configuration.

The present formulation follows that for surface waves in a homogeneous fluid (Miles 1976, hereinafter referred to as I), which has proved useful for various investigations of nonlinear stability and chaotic-motion problems (e.g. Miles 1984*a*, *b*). The free-surface displacement in that problem may be posited in the form $\eta(\boldsymbol{x}, t) = q_n(t)\,\psi_n(\boldsymbol{x})$, where $\boldsymbol{x} \equiv (x_1, x_2)$ is the horizontal coordinate, $\{\psi_n(\boldsymbol{x})\}$ is a complete set of normal modes, $\boldsymbol{q} \equiv \{q_n(t)\}$ is the corresponding set of generalized coordinates, and the summation convention is implicit. The Lagrangian then has the form

$$L = \tfrac{1}{2}\rho S[a_{mn}(\boldsymbol{q})\,\dot{q}_m\,\dot{q}_n - g\delta_{mn}\,q_m\,q_n], \tag{1.1}$$

where S is the cross-sectional area of the cylinder and

$$a_{mn} = \delta_{mn}\,a_n + a_{lmn}\,q_l + \tfrac{1}{2}a_{jlmn}\,q_j\,q_l + \ldots \tag{1.2}$$

is an inertial matrix (with the dimensions of length). The truncation $a_{mn} \doteqdot \delta_{mn}\,a_n$ implies (through Hamilton's principle) a set of uncoupled differential equations for the $q_n(t)$, the solution of which completes the classical description (in particular, determines the natural frequencies) of small oscillations. The two-term truncation $a_{mn} = \delta_{mn}\,a_n + a_{lmn}\,q_l$ yields a set of quadratically coupled differential equations, which typically provide a second-order description of the nonlinear surface waves. It is necessary to proceed to the three-term truncation displayed in (1.2) to determine the amplitude dependence (which is quadratic in this approximation) of the resonant frequencies.

This formulation for a homogeneous fluid may be extended to a stratified fluid that is modelled by a sequence of N layers, in each of which the density is constant. A flow started from rest then is irrotational, and a velocity potential exists, *within* each layer [the overall flow is rotational, with the vorticity being concentrated at the interfaces; cf. Lamb 1932, §231]. In §2, I express the Lagrangian for such a layer as a functional of the velocity potential ϕ, and the displacements of the upper and lower boundaries (which may be rigid or free surfaces or interfaces with adjacent layers) $\eta_\pm$, where the subscript $\pm$ signifies *upper/lower*. In §3, I posit the Fourier expansions $\eta_\pm = q_n^\pm(t)\,\psi_n(\boldsymbol{x})$ and a corresponding expansion of ϕ and determine the coefficients in the latter expansion in terms of $q_n^\pm$ and $\dot{q}_n^\pm$ through the requirement that the Lagrangian be stationary with respect to variations of ϕ. The Lagrangian for a single layer then may be placed in the form $\rho S L_1(\boldsymbol{q}_+, \dot{\boldsymbol{q}}_+, \boldsymbol{q}_-, \dot{\boldsymbol{q}}_-; d)$, wherein d and S are respectively the thickness and area of the undisturbed layer. The corresponding Lagrangian for an N-layer fluid in which the density, thickness and generalized coordinates of the upper and lower boundaries of the νth layer are ρ_ν, d_ν, $\boldsymbol{q}_\nu$ and $\boldsymbol{q}_{\nu-1}$, respectively, is given by

$$\mathscr{L} = S \sum_{\nu=1}^{N} \rho_\nu L_1(\boldsymbol{q}_\nu, \dot{\boldsymbol{q}}_\nu, \boldsymbol{q}_{\nu-1}, \dot{\boldsymbol{q}}_{\nu-1}, d_\nu), \tag{1.3}$$

It is worth emphasizing that (1.3) does not require the Boussinesq approximation, although this approximation is subsequently invoked in some of the examples and in the limiting case of continuous stratification.

The configuration of a single interface with rigid boundaries (the simplest, non-trivial two-layer problem) is obtained by setting $N = 2$, $\boldsymbol{q}_0 = \boldsymbol{q}_2 = 0$, $\boldsymbol{q}_1 \equiv \boldsymbol{q}$, $\rho_{1,2} \equiv \rho_\mp$ and $d_{1,2} \equiv d_\mp$. I consider three-dimensional standing and progressive waves for this configuration in §§4 and 5, respectively, and reproduce the two-dimensional results of Hunt (1961) and Thorpe (1968*a*, *b*) after allowing for (what appear to be) typographical errors therein. Explicit, three-dimensional examples are straightforward in principle but algebraically complicated.

I originally undertook the present investigation in connection with the conjecture of Mahony & Smith (1972) that a spatial resonance might occur between surface and internal waves of comparable wavenumbers k and $2k$ respectively, and disparate natural frequencies, ω_s and ω_i, where $(\omega_i/\omega_s)^2 = O(\epsilon)$ and ϵ is a measure of the stratification. It turns out, however, that the coupling between these two modes is $O(\epsilon)$, rather than $O(1)$ as in Mahony & Smith's model problem of spatial resonance between aerial and surface waves, and the putative resonance is either impossible or realized only at higher order (in which case it presumably would be of limited geophysical interest). I therefore have relegated this particular application to the Appendix.

The Lagrangian for a continuously stratified fluid, in which $\eta(\boldsymbol{x}, y, t) = q_n(y, t)\, \psi_n(\boldsymbol{x})$ and y is a Lagrangian (rather than an Eulerian) coordinate, may be obtained by setting $d_\nu = \delta$, $y = \nu\delta$, $\rho_\nu = \rho(y)$, and $D = N\delta$ in (1.3) and then letting $\delta \downarrow 0$ with D fixed. I carry out this limit in §6 for a Boussinesq fluid (in which the inertial effects of stratification are neglected), apply the result to progressive waves, and obtain second-order approximations to η and the dispersion relation for any density profile for which the (linear) vertical structure problem can be solved. Considering the particular cases of finite depth with a linear density profile and infinite depth with a hyperbolic tangent density profile, I obtain Thorpe's (1968*b*) approximations to η and the corresponding dispersion relations (which appear to be new).

It should be emphasized that, although the Lagrangian formulation appreciably simplifies the algebra for standing- and progressive-wave problems in which the time dependence is harmonic, it is most valuable in attacking problems of stability and chaotic motion in which a basic harmonic time dependence is slowly modulated, with the typical representation

$$q_n(t) = a_n(\tau) \cos n\omega t + b_n(\tau) \sin n\omega t, \tag{1.4}$$

where τ is a slow time. Averaging the Lagrangian over the fast period $2\pi/\omega$ then leads, through the invocation of Hamilton's principle, to a set of evolution equations for the a_n and b_n. Examples are parametrically excited solitary waves in a long channel (Miles 1984*a*), forced, internally resonant surface waves in a circular cylinder (Miles 1984*b*), and stratified shear flow over a sinusoidal bottom (Miles 1986).

2. Lagrangian for single layer

We consider the irrotational motion of a layer of incompressible fluid that is bounded laterally by a rigid cylinder of cross-section S and above and below by the interfaces $y = \frac{1}{2}d + \eta_+(\boldsymbol{x}, t)$ and $y = -\frac{1}{2}d + \eta_-(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ and y are horizontal and vertical coordinates. The Lagrangian for this motion may be posed in the form (Luke 1967; Whitham 1974, pp. 435–6)

$$\mathscr{L} = \iint \mathrm{d}S \int_{y_-}^{y_+} p \, \mathrm{d}y \quad (y_\pm \equiv \pm\tfrac{1}{2}d + \eta_\pm), \tag{2.1}$$

where p is the pressure and, here and subsequently, the alternative signs are vertically ordered. Invoking the assumptions of incompressibility and irrotationality, we have

$$p = p_0 - \rho(\phi_t + \tfrac{1}{2}\nabla\phi \cdot \nabla\phi + gy), \tag{2.2}$$

where p_0 is the (static) equilibrium pressure at $y = 0$, ρ is the density, $\phi(\boldsymbol{x}, y, t)$ is a velocity potential (velocity $= \nabla\phi$) that satisfies

$$\nabla^2\phi = 0 \tag{2.3}$$

and

$$\boldsymbol{n} \cdot \nabla\phi = 0 \quad \text{on } \partial S, \tag{2.4}$$

and $\phi_t \equiv \partial\phi/\partial t \equiv \partial_t \phi$. We also impose the constraint

$$\iint \eta_+ \, \mathrm{d}S = \iint \eta_- \, \mathrm{d}S = 0. \tag{2.5}$$

Conservation of volume implies only that these last two integrals are equal, but we ultimately assume that $\eta = 0$ at the lower boundary of an N-layered fluid, by virtue of which (2.5) holds at each of the interfaces. If the lower boundary is rigid but of variable relief, the origin of y may be chosen to make the mean value of η vanish.

Substituting (2.2) into (2.1) and invoking (2.5) and the identities

$$\int_{y_-}^{y_+} \phi_t \, \mathrm{d}y = \partial_t \int_{y_-}^{y_+} \phi \, \mathrm{d}y - (\phi\eta_t)|_{y_-}^{y_+} \tag{2.6}$$

and

$$\iint \mathrm{d}S \int_{y_-}^{y_+} (\nabla\phi)^2 \, \mathrm{d}y = \iint [\phi(\phi_y - \nabla\eta \cdot \nabla\phi)]|_{y_-}^{y_+} \, \mathrm{d}S, \tag{2.7}$$

which follows from Green's theorem, (2.3) and (2.4), we transform (2.1) to

$$\mathscr{L} = \rho S L + \mathscr{D}, \quad \mathscr{D} = p_0 S d - \rho\partial_t \iiint_{y_-}^{y_+} \phi \, \mathrm{d}S \, \mathrm{d}y, \tag{2.8a, b}$$

where

$$L = S^{-1} \iint [\phi\eta_t - \tfrac{1}{2}\phi(\phi_y - \nabla\eta \cdot \nabla\phi) - \tfrac{1}{2}g\eta^2]|_{y_-}^{y_+} \, \mathrm{d}S. \tag{2.9}$$

The functional $\mathscr{D}$ makes no contribution to the variation of the action, $\int \mathscr{L} \, \mathrm{d}t$, and therefore is of no further interest. The remaining Lagrangian, $\rho S L$, is a functional of ϕ, η_+ and η_-.

3. Fourier expansion

We now posit the Fourier expansions

$$\eta_{\pm} = q_n^{\pm}(t)\,\psi_n(\boldsymbol{x}), \tag{3.1}$$

where, here and subsequently except as noted, the repeated index n is summed over the Fourier spectrum, and $\{\psi_n\}$ is the complete orthogonal set determined by

$$(\nabla^2+k^2)\,\psi_n = 0, \quad \boldsymbol{n}\cdot\nabla\psi_n = 0 \quad \text{on } \partial S, \quad \iint \psi_m\,\psi_n\,\mathrm{d}S = \delta_{mn}\,S \qquad (3.2\,a,\,b,\,c)$$

for $k = k_n$. We posit the corresponding expansion for the velocity potential in the form

$$\phi = \left[\frac{\phi_n^+(t)\,\cosh k_n(y+\tfrac{1}{2}d)+\phi_n^-(t)\,\cosh k_n(y-\tfrac{1}{2}d)}{\cosh k_n\,d}\right]\psi_n(\boldsymbol{x}). \tag{3.3}$$

It follows from (2.5) that $q_0^{\pm} = 0$, where $\psi_0 = 1$ is the constant member of the set $\{\psi_n\}$, but it does not follow that $\phi_0^{\pm} = 0$ (cf. I, §5).

Substituting (3.1) and (3.3) into (2.9), expanding ϕ about $y = y_{\pm}$, and carrying out the integrations with respect to $\boldsymbol{x}$, we obtain the quartic approximation

$$\begin{aligned} L = \delta_{mn}[&(\dot q_m^+ - S_m\,\dot q_m^-)\,\phi_n^+ + (S_m\,\dot q_m^+ - \dot q_m^-)\,\phi_n^- - \tfrac{1}{2}g(q_m^+\,q_n^+ - q_m^-\,q_n^-) \\ &-\tfrac{1}{2}\ell_n(\phi_m^+\,\phi_n^+ + 2S_n\,\phi_m^+\,\phi_n^- + \phi_m^-\,\phi_n^-)] + \ell_n\,C_{lmn}(q_l^+\,\dot q_m^+\,\phi_n^+ + q_l^-\,\dot q_m^-\,\phi_n^-) \\ &+\tfrac{1}{2}\ell_n^2\,C_{jlmn}[q_j^+\,q_l^+\dot q_m^+(\phi_n^+ + S_n\,\phi_n^-) - q_j^-\,q_l^-\,\dot q_m^-(S_n\,\phi_n^+ + \phi_n^-)] \\ &-\tfrac{1}{2}[\ell_{lmn}\,q_l^+ - D_{lmn}\,S_m\,S_n\,q_l^- + \tfrac{1}{2}\ell_{jlmn}\,q_j^+\,q_l^+]\,\phi_m^+\,\phi_n^+ \\ &-\tfrac{1}{2}[-\ell_{lmn}\,q_l^- + D_{lmn}\,S_m\,S_n\,q_l^+ + \tfrac{1}{2}\ell_{jlmn}\,q_j^-\,q_l^-]\,\phi_m^-\,\phi_n^- \\ &-\tfrac{1}{2}[2D_{lmn}(S_m\,q_l^+ - S_n\,q_l^-) + \tfrac{1}{2}\ell_{jlmn}(S_m\,q_j^+\,q_l^+ + S_n\,q_j^-\,q_l^-]\,\phi_m^-\,\phi_n^+, \end{aligned} \tag{3.4}$$

where (the summation convention does not apply in any of (3.5a)–(3.8b)

$$C_{lmn} = S^{-1}\iint \psi_l\,\psi_m\,\psi_n\,\mathrm{d}S, \quad C_{jlmn} = S^{-1}\iint \psi_j\,\psi_l\,\psi_m\,\psi_n\,\mathrm{d}S, \qquad (3.5\,a,\,b)$$

$$D_{lmn} = S^{-1}\iint \psi_l\,\nabla\psi_m\cdot\nabla\psi_n\,\mathrm{d}S, \quad D_{jlmn} = S^{-1}\iint \psi_j\,\psi_l\,\nabla\psi_m\cdot\nabla\psi_n\,\mathrm{d}S, \qquad (3.6\,a,\,b)$$

$$\ell_n = k_n\,\tanh k_n\,d \equiv 1/a_n, \quad S_n = \operatorname{sech} k_n\,d, \qquad (3.7\,a,\,b)$$

$$\ell_{lmn} = C_{lmn}\,\ell_m\,\ell_n + D_{lmn}, \qquad (3.8\,a)$$

$$\ell_{jlmn} = C_{jlmn}(k_m^2\,\ell_n + k_n^2\,\ell_m) + (\ell_m + \ell_n)\,D_{jlmn}, \qquad (3.8\,b)$$

and the various coefficients have been simplified with the aid of the identities given in I (A 8, 9).

The requirement (Hamilton's principle) that L be stationary with respect to independent variations of the $\phi_n^{\pm}$ yields a set of equations that may be solved for $\phi_n^{\pm}$ in terms of the $q_n^{\pm}$ and $\dot q_n^{\pm}$ by iteration, starting from the first approximation $\ell_n\,\phi_n^{\pm} = \pm\dot q_n^{\pm}$ (n not summed). Moreover, the error in L is of the order of the square of the error in $\phi_n^{\pm}$ by virtue of stationarity; accordingly, it suffices for the quartic approximation to L to have the quadratic approximation to $\phi_n^{\pm}$, which is found to be

$$\phi_n^{\pm} = a_n(\pm\dot q_n^{\pm} - r_n^{\pm}), \quad r_n^{\pm} = D_{lmn}\,a_m\,q_l^{\pm}(\dot q_m^{\pm} - S_m\,\dot q_m^{\mp}) \quad (n \text{ not summed}). \qquad (3.9\,a,\,b)$$

The substitution of (3.9) into (3.4), followed by quartic truncation, yields (after extensive manipulation)

$$L \equiv L_1(\boldsymbol{q}_+, \dot{\boldsymbol{q}}_+, \boldsymbol{q}_-, \dot{\boldsymbol{q}}_-; d) = \tfrac{1}{2}\delta_{mn}[a_n(\dot{q}_m^+ \dot{q}_n^+ - 2S_n \dot{q}_m^+ \dot{q}_n^- + \dot{q}_m^- \dot{q}_n^-) - g(q_m^+ q_n^+ - q_m^- q_n^-)]$$
$$+ \tfrac{1}{2}C_{lmn}(q_l^+ \dot{q}_m^+ \dot{q}_n^+ - q_l^- \dot{q}_m^- \dot{q}_n^-)$$
$$- \tfrac{1}{2}D_{lmn}\, a_m\, a_n[q_l^+(\dot{q}_m^+ - S_m \dot{q}_m^-)(\dot{q}_n^+ - S_n \dot{q}_n^-) - q_l^-(\dot{q}_m^- - S_m \dot{q}_m^+)(\dot{q}_n^- - S_n \dot{q}_n^+)] + L_4, \tag{3.10a}$$

wherein $\boldsymbol{q}_\pm \equiv \{q_n^\pm\}$, the functional L_1 is defined as in (1.3), and

$$L_4 = \tfrac{1}{2}\delta_{mn}\, a_n(r_m^+ r_n^+ + 2S_n r_m^+ r_n^- + r_m^- r_n^-)$$
$$+ \tfrac{1}{4}C_{jlmn}\, k_m^2\, a_m[q_j^+ q_l^+(S_n \dot{q}_m^+ \dot{q}_n^- - S_m \dot{q}_m^- \dot{q}_n^+) + q_j^- q_l^-(S_n \dot{q}_m^- \dot{q}_n^+ - S_m \dot{q}_m^+ \dot{q}_n^-)]$$
$$- \tfrac{1}{4}D_{jlmn}(a_m + a_n)[q_j^+ q_l^+(\dot{q}_m^+ - S_m \dot{q}_m^-)\dot{q}_n^+ + q_j^- q_l^-(\dot{q}_m^- - S_m \dot{q}_m^+)\dot{q}_n^-] \tag{3.10b}$$

comprises the quartic terms. We remark that, in many problems for which a quartic approximation to L is required, only a single mode need be retained in the calculation of L_4; see e.g. (4.8) below.

The surface-wave formulation of I for a single layer is recovered by setting $q_n^- = 0$ (level bottom), which implies $\phi_n^- = 0$.

4. Interfacial waves

The configuration of a single interface with rigid boundaries (the simplest, non-trivial two-layer problem) is obtained by setting $N = 2$, $\boldsymbol{q}_0 = \boldsymbol{q}_2 = 0$, $\boldsymbol{q}_1 \equiv \boldsymbol{q}$, $\rho_{1,2} \equiv \rho_\mp$ and $d_{1,2} \equiv d_\mp$ (where the mnemonic subscript $\pm$ now designates the upper/lower layer) in (1.3), which then reduces to

$$\mathscr{L} = S[\rho_- L_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, 0, 0; d_-) + \rho_+ L_1(0, 0, \boldsymbol{q}, \dot{\boldsymbol{q}}; d_+)]. \tag{4.1}$$

Combining (4.1) with (3.10), we obtain [cf. (1.1), (1.2)]

$$\mathscr{L} = \tfrac{1}{2}\rho_* S[(\delta_{mn}\, a_n + a_{lmn}\, q_l + \tfrac{1}{2}a_{jlmn}\, q_j\, q_l)\dot{q}_m \dot{q}_n - \delta_{mn}\, a_n\, \omega_n^2\, q_m\, q_n], \tag{4.2}$$

where the reference density ρ_* is arbitrary,

$$a_n = \frac{\rho_+ a_n^+ + \rho_- a_n^-}{\rho_*}, \quad a_{lmn} = \frac{\rho_- a_{lmn}^- - \rho_+ a_{lmn}^+}{\rho_*}, \quad a_{jlmn} = \frac{\rho_+ a_{jlmn}^+ + \rho_- a_{jlmn}^-}{\rho_*}, \tag{4.3a, b, c}$$

$a_n^\pm$ is given by (3.7a) with $d = d_\pm$ therein,

$$a_{lmn}^\pm = C_{lmn} - D_{lmn}\, a_m^\pm\, a_n^\pm, \quad a_{jlmn}^\pm = 2D_{jmi}\, D_{lni}\, a_i^\pm\, a_m^\pm\, a_n^\pm - D_{jlmn}(a_m^\pm + a_n^\pm), \tag{4.4a, b}$$

the summation convention does not apply in (4.3) and (4.4), C_{lmn}, D_{lmn}, C_{jlmn} and D_{jlmn} are given by (3.5) and (3.6), and

$$\omega_n^2 = \frac{(\rho_- - \rho_+)g}{\rho_+ a_n^+ + \rho_- a_n^-}. \tag{4.5}$$

The linear approximation (cf. Lamb 1932, §231), which follows from the quadratic approximation to $\mathscr{L}$, yields uncoupled oscillations (the normal modes) with the frequencies ω_n. Now suppose that $q_n = \delta_{1n} A_1 \cos \omega t$ and $\omega = \omega_1$ describe the first approximation to the nonlinear oscillations governed by (4.2) for a particular mode

($n = 1$ need not imply the dominant mode). Then, proceeding as in I, §6, we seek a second approximation in the form

$$q_n = \delta_{1n} A_1 \cos \omega t + A_{n0} + A_{n2} \cos 2\omega t, \tag{4.6}$$

where A_{n0}, $A_{n2} = O(A_1^2)$. Substituting (4.6) into (4.2), averaging $\mathscr{L}$ over the period $2\pi/\omega$, requiring the average Lagrangian $\langle \mathscr{L} \rangle$ to be stationary with respect to each of A_1, A_{n0} and A_{n2} and approximating ω by ω_1 except in the term $\omega^2 - \omega_1^2$, we obtain

$$A_{n0} = \frac{1}{4}\left(\frac{a_{n11}}{a_1}\right) A_1^2, \quad A_{n2} = -\frac{1}{4}\left(\frac{4a_{11n} - a_{n11}}{4a_n - a_1}\right) A_1^2 \quad (n \text{ not summed}) \tag{4.7a, b}$$

and
$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + \frac{1}{4}\left(\frac{A_1}{a_1}\right)^2 \left[\frac{a_1(4a_{11n} - a_{n11})^2}{2(4a_n - a_1)} - a_{n11} a_{n11} - a_1 a_{1111}\right], \tag{4.8}$$

where ω_1^2 is given by (4.5).

The results (4.7) and (4.8) are valid for three-dimensional (two-dimensional on the interface) waves in any container for which the normal mode problem (3.2) can be solved; however, the algebra for specific three-dimensional problems is quite involved, and we therefore consider as examples only two-dimensional waves in a basin of length π/k, for which

$$\psi_n = 2^{\frac{1}{2}} \cos k_n x, \quad k_n = nk \quad (n = 1, 2, \ldots). \tag{4.9a, b}$$

We then find that only those terms for which $n = 2$ in (4.7) and (4.8) make non-zero contributions (since $C_{11n} = 0$ for $n \neq 2$). Replacing A_1 by $A/\surd 2$, choosing $\rho_* \equiv \rho_+$, introducing

$$\epsilon \equiv \frac{\rho_- - \rho_+}{\rho_+}, \tag{4.10}$$

and letting $\epsilon \downarrow 0$ (the Boussinesq approximation), we obtain

$$\omega_1^2 = \frac{\epsilon g k T_+ T_-}{T_+ + T_-} \quad (T_\pm \equiv \tanh k d_\pm), \tag{4.11}$$

$$\eta = A \cos \omega t \cos kx + \tfrac{1}{8} k A^2 \left(\frac{T_+ - T_-}{T_+^2 T_-^2}\right)(T_+ T_- + 3 \cos 2\omega t) \cos 2kx, \tag{4.12}$$

and
$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + k^2 A^2 \left[\frac{9(T_+ - T_-)^2 - 2T_+ T_-(3T_+^2 - 4T_+ T_- + 3T_-^2)}{32 T_+^3 T_-^3}\right], \tag{4.13}$$

which are equivalent to Thorpe's (1968*a*) results. The approximation (4.12) fails if $d_+ \doteqdot d_-$, in which case the amplitude of the second harmonic is $O(\epsilon k A^2)$.

5. Progressive waves

We now suppose that the fluid is laterally unbounded (or, more precisely, of lateral dimensions large compared with $1/k$), replace the eigenfunctions of (3.2) by

$$\psi_n = e^{i\boldsymbol{k}_n \cdot \boldsymbol{x}}, \quad \psi_{\bar{n}} \equiv \psi_n^* = e^{-i\boldsymbol{k}_n \cdot \boldsymbol{x}}, \tag{5.1a, b}$$

where the asterisk signifies complex-conjugation, and extend the summations in (3.1) and (3.3) over both ψ_n and $\psi_{\bar{n}}$ with $q_{\bar{n}} \equiv q_n^*$ and $k_n \equiv |\boldsymbol{k}_n|$. The spatial average of

L/ρ (after discarding the counterpart of $\hat{L}$) then is given by (3.10) with the correlation coefficients therein replaced by

$$\delta_{mn} = \begin{matrix} 1 \\ 0 \end{matrix} \quad \text{for } \boldsymbol{k}_m + \boldsymbol{k}_n \begin{matrix} = \\ \neq \end{matrix} 0, \tag{5.2}$$

$$C_{lmn} = \begin{matrix} 1 \\ 0 \end{matrix} \quad \text{for } \boldsymbol{k}_l + \boldsymbol{k}_m + \boldsymbol{k}_n \begin{matrix} = \\ \neq \end{matrix} 0, \quad C_{jlmn} = \begin{matrix} 1 \\ 0 \end{matrix} \quad \text{for } \boldsymbol{k}_j + \boldsymbol{k}_l + \boldsymbol{k}_m + \boldsymbol{k}_n \begin{matrix} = \\ \neq \end{matrix} 0, \tag{5.3a, b}$$

$$D_{lmn} = -C_{lmn}\, \boldsymbol{k}_m \cdot \boldsymbol{k}_n, \quad D_{jlmn} = -C_{jlmn}\, \boldsymbol{k}_m \cdot \boldsymbol{k}_n, \tag{5.4a, b}$$

and D_{lmn} replaced by $D_{lm\bar{n}}$ in (3.9*b*).

Considering, for example, the two-layer configuration of §4 and a progressive wave for which

$$\psi_n = e^{inkx} \quad (n = \pm 1, \pm 2, \ldots) \tag{5.5}$$

and $q_1 = A_1 \exp(-i\omega t)$ and $\omega = \omega_1$ (4.5) describe the first approximation, we posit the counterpart of (4.6) in the form

$$q_n = A_n\, e^{-in\omega t} \quad (n = \pm 1, \pm 2). \tag{5.6}$$

The Lagrangian retains the form (4.2), with D_{jml} replaced by $D_{jm\bar{l}}$ in (4.4*b*). Substituting (5.6) into (4.2) and invoking (5.2)–(5.4), which imply $a_{lmn} = 0$ unless $l+m+n = 0$ and $a_{jlmn} = 0$ unless $j+l+m+n = 0$, we obtain†

$$L \equiv (\rho_* S)^{-1} \mathscr{L} = a_1(\omega^2 - \omega_1^2)\, A_1 A_1^* + a_2(4\omega^2 - \omega_2^2)\, A_2 A_2^*$$
$$+ (2a_{11\bar{2}} - \tfrac{1}{2} a_{\bar{2}11})\, \omega^2 (A_1^2 A_2^* + A_1^{*2} A_2) + \tfrac{1}{2}(a_{1\bar{1}1\bar{1}} + a_{1\bar{1}\bar{1}1} - a_{11\bar{1}\bar{1}})\, \omega^2 A_1^2 A_1^{*2}. \tag{5.7}$$

Requiring $\mathscr{L}$ to be stationary with respect to independent variations of A_1^* and A_2^*, approximating ω by ω_1 except in the term $\omega^2 - \omega_1^2$, and solving the resulting equations for A_2 and ω^2, we obtain

$$A_2 = -\frac{1}{2}\left(\frac{4a_{11\bar{2}} - a_{2\bar{1}\bar{1}}}{4a_2 - a_1}\right) A_1^2 \tag{5.8}$$

and

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + \left[\frac{(4a_{11\bar{2}} - a_{2\bar{1}\bar{1}})^2}{2a_1(4a_2 - a_1)} + \left(\frac{a_{11\bar{1}\bar{1}} - a_{1\bar{1}1\bar{1}} - a_{1\bar{1}\bar{1}1}}{a_1}\right)\right] |A_1|^2 \tag{5.9}$$

as the counterparts of (4.7) and (4.8). Evaluating the coefficients through (4.3), (4.4), (5.3) and (5.4), replacing A_1 by $\tfrac{1}{2}A$, invoking (4.10), and letting $\epsilon \downarrow 0$, we obtain

$$\eta = A \cos(kx - \omega t) + \tfrac{3}{4} k A^2 \left(\frac{T_+ - T_-}{T_+^2 T_-^2}\right) \cos 2(kx - \omega t) \tag{5.10}$$

and

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + k^2 A^2 \left[1 - \frac{1}{2}\left(\frac{1}{T_+^2} - \frac{1}{T_+ T_-} + \frac{1}{T_-^2}\right) + \frac{9}{8}\,\frac{(T_+ - T_-)^2}{T_+^3 T_-^3}\right], \tag{5.11}$$

which are in agreement with the results of Hunt (1961) and Thorpe (1968*b*) after correcting typographical errors therein.‡ It is worth noting that (5.8) and (5.9) do not rest on the Boussinesq approximation whereas (5.10) and (5.11) do.

† Note that (5.7) is automatically the temporal average of the Lagrangian for a simple progressive wave but that it would be necessary to average over the period $2\pi/\omega$ if the amplitudes were slowly varying functions of x and t (cf. Simmons 1969).

‡ The sign of ma_2 should be reversed in Hunt's expression for c^2 (p. 525). $(3-T_2)^2$ and $(3-T_1)^2$ should be replaced by $(3-T_2^2)$ and $(3-T_1^2)$ in Thorpe's (2.1.2), $(3-T^2)$ and $(\rho_1+\rho_2)$ should be replaced by $(3-T^2)^2$ and $(\rho_1+\rho_2)^2$ in his (2.1.6), and $8T_i$ should be replaced by $8T_i^3$ in his (2.1.7). Dr Thorpe agrees with these corrections.

6. Continuous stratification

We obtain the Lagrangian for a continuously stratified fluid, in which

$$\eta(\boldsymbol{x}, y, t) = q_n(y, t)\, \psi_n(\boldsymbol{x}), \tag{6.1}$$

by setting $d_\nu \equiv d$, $y \equiv \nu d$ and $D \equiv Nd$ in (1.3) and then letting $d \downarrow 0$ with D fixed:

$$\mathscr{L} = S \lim_{d\downarrow 0} \sum_{\nu=1}^{D/d} \rho_\nu L_1(\boldsymbol{q}_\nu, \dot{\boldsymbol{q}}_\nu, \boldsymbol{q}_{\nu-1}, \dot{\boldsymbol{q}}_{\nu-1}; d). \tag{6.2}$$

The Boussinesq approximation implies $\rho_\nu \approx$ constant $\equiv \rho_*$ except in the potential-energy term, which transforms according to

$$\sum_{\nu=1}^{N} \rho_\nu(\boldsymbol{q}_\nu^2 - \boldsymbol{q}_{\nu-1}^2) \to \int_0^D \rho(y)(\partial \boldsymbol{q}^2/\partial y)\,\mathrm{d}y = -\int_0^D \rho'(y)\,\boldsymbol{q}^2\,\mathrm{d}y + (\rho\boldsymbol{q}^2)_D, \tag{6.3}$$

wherein y appears as a Lagrangian coordinate†, $\rho' \equiv \mathrm{d}\rho/\mathrm{d}y$, the subscript D implies evaluation at the upper boundary, and we have assumed $\boldsymbol{q} = 0$ at the lower boundary. Transforming the remaining terms in (3.10) with the aid of the limits

$$a_n \to (dk_n^2)^{-1} \equiv d^{-1}\lambda_n^2, \quad S_n \to 1 - \tfrac{1}{2}d^2k_n^2, \tag{6.4a, b}$$

$$q_n^+ - q_n^- \to d\frac{\partial q_n}{\partial y} \equiv dq_n', \tag{6.5}$$

and introducing

$$N^2 \equiv -g\rho'/\rho \tag{6.6}$$

for the square of the buoyancy frequency (N no longer appears as an index), we obtain

$$\begin{aligned} L \equiv (\rho_* S)^{-1}\mathscr{L} = \frac{1}{2}\int_0^D [&\delta_{mn}(\lambda_m\lambda_n\dot{q}_m'\dot{q}_n' + \dot{q}_m\dot{q}_n - N^2 q_m q_n) \\ &- D_{lmn}(\lambda_m^2\lambda_n^2 q_l'\dot{q}_m'\dot{q}_n' + 2\lambda_m^2 q_l\dot{q}_m'\dot{q}_n) + 2\overline{L}_4]\,\mathrm{d}y \\ &+ \tfrac{1}{2}(-g\delta_{mn}q_m q_n + C_{lmn}q_l\dot{q}_m\dot{q}_n)_D, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \overline{L}_4 = \tfrac{1}{2}\{&D_{jmi}D_{lni}[\lambda_i^2(\lambda_m^2 q_j'\dot{q}_m' + q_j\dot{q}_m)(\lambda_n^2 q_l'\dot{q}_n' + q_l\dot{q}_n) + \lambda_m^2\lambda_n^2 q_j q_l\dot{q}_m'\dot{q}_n'] \\ &- \tfrac{1}{2}D_{jlmn}(\lambda_m^2 + \lambda_n^2)[(q_j q_l\dot{q}_m)'\dot{q}_n' + k_m^2 q_j q_l\dot{q}_m\dot{q}_n]\}. \end{aligned} \tag{6.8}$$

The partially integrated terms may be neglected for the internal waves, for which $q_n \doteqdot 0$ at $y = D$. [It can be shown that the equations of motion implied by (6.7) through Hamilton's principle are equivalent, after Fourier inversion, to Phillips's (1977) equation (5.2.4). It follows from this equivalence that the assumption of irrotationality in the individual layers of the layered model of a continuously stratified fluid does not impose any qualitative restriction on the rotational solutions in the continuous fluid, at least in the absence of non-conservative body forces.]

Consider, for example, a two-dimensional progressive wave for which the ψ_n are given by (5.5) and

$$q_n(y, t) = Q_n(y)\,\mathrm{e}^{-\mathrm{i}n\omega t} \quad (n \text{ not summed}). \tag{6.9}$$

The complex amplitudes may be expanded in the eigenfunctions of the linear problem, f_{mn}, according to

$$Q_n(y) = A_{mn}f_{mn}(y) \quad (n \text{ not summed}), \tag{6.10}$$

† Milder (1982) designates the equivalent of y as an 'iso-pycnal' coordinate in his Lagrangian formulation for a continuously stratified fluid.

where the f_{mn} are determined by the Sturm–Liouville problem

$$f''_{mn}+k_n^2\left(\frac{N^2}{\omega_{mn}^2}-1\right)f_{mn}=0 \quad (m,\, n \text{ not summed}), \tag{6.11a}$$

$$f_{mn}=0 \quad (y=0, D), \tag{6.11b}$$

and are orthogonal and normalized according to

$$\int_0^D f_{ln} f_{mn} N^2 \,\mathrm{d}y=\delta_{lm} g', \quad g'=\int_0^D N^2 \,\mathrm{d}y = g\left[\frac{\rho(0)-\rho(D)}{\rho_*}\right]. \tag{6.12a, b}$$

The natural frequencies ω_{mn} are the eigenvalues for prescribed $k_n \equiv nk$ (cf. Phillips 1977, §5.2). We also note that

$$\int_0^D (\lambda_n^2 f'_{ln} f'_{mn}+f_{ln} f_{mn})\,\mathrm{d}y=\frac{\delta_{lm} g'}{\omega_{mn}^2}. \tag{6.13}$$

Substituting (6.9) into (6.7), choosing $A_{m1}\equiv \delta_{1m} A_1$, truncating the expansion in the ψ_n at $n=2$, as in §5, and invoking the orthogonality relations (6.12) and (6.13), we obtain

$$L = a_1(\omega^2-\omega_1^2) A_1 A_1^* + a_{m2}(4\omega^2-\omega_{m2}^2) A_{m2} A_{m2}^* + a_{11m2}\,\omega^2 (A_1^2 A_{m2}^* + A_1^{*2} A_{m2}) + \tfrac{1}{2} a_{1111}\,\omega^2 A_1^2 A_1^{*2}, \tag{6.14}$$

where

$$a_1=\frac{g'}{\omega_1^2}, \quad a_{m2}=\frac{g'}{\omega_{m2}^2}, \tag{6.15a}$$

$$a_{11m2}=-\frac{3}{2}\int_0^D (\lambda_1^2 f_1'^2-f_1^2) f'_{m2}\,\mathrm{d}y, \tag{6.15b}$$

$$a_{1111}=2\int_0^D (\lambda_1^2 f_1'^4-3f_1^2 f_1'^2-2k_1^2 f_1^4)\,\mathrm{d}y, \tag{6.15c}$$

and $f_1\equiv f_{11}$. Requiring L to be stationary with respect to independent variations of A_1^* and A_2^*, as in §5, we obtain

$$A_{m2}=-\left(\frac{a_{11m2}}{4a_{m2}-a_1}\right) A_1^2 \tag{6.16}$$

and

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1+\left[\frac{2a_{11m2}^2}{a_1(4a_{m2}-a_1)}-\frac{a_{1111}}{a_1}\right]|A_1|^2, \tag{6.17}$$

which provide the quadratic approximations to η and the dispersion relation for any density profile for which the (linear) vertical structure problem (6.11) can be solved.

Turning to the special case of constant N^2, for which

$$N^2=\frac{g'}{D}, \quad f_{mn}=\sqrt{2}\sin\left(\frac{m\pi y}{D}\right), \quad \omega_{mn}^2=\frac{N^2}{1+(m\theta/n)^2}, \tag{6.18a, b, c}$$

where

$$\theta\equiv\frac{\pi}{kD}, \tag{6.19}$$

we have

$$a_1=(1+\theta^2)D, \quad a_{mn}=\left[1+\left(\frac{m\theta}{n}\right)^2\right]D, \tag{6.20a, b}$$

$$a_{11m2}=-3.2^{-\frac{1}{2}}\pi\delta_{m2}(\theta^2+1), \quad a_{1111}=3k^2D(\theta^2+1)(\theta^2-2). \tag{6.20c, d}$$

Substituting (6.20) into (6.16) and (6.17), introducing $A \equiv 2^{\frac{3}{2}} A_1$, and invoking (5.1), (6.9) and (6.10), we obtain

$$\eta = A \cos(kx-\omega t) \sin\left(\frac{\pi y}{D}\right) + \tfrac{1}{4}\pi \frac{A^2}{D} \cos 2(kx-\omega t) \sin\left(\frac{2\pi y}{D}\right) \tag{6.21}$$

and

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + \tfrac{3}{4} k^2 A^2. \tag{6.22}$$

The approximation (6.21) is equivalent to Thorpe's (1968*b*) equation (3.3.4) if his $f_3 = 0$ and his z is replaced by his z_0†, which is equivalent to the present Lagrangian coordinate y. The approximation (6.22), which appears to be new, presumably owes its simplicity (in particular, its independence of D) to the uniformity of N; cf. (6.25) below.

The corresponding results for the dominant mode of the profile

$$N^2 = \frac{1}{2}\frac{g'}{h} \operatorname{sech}^2\left(\frac{y}{h}\right) \quad (-\infty < y < \infty) \tag{6.23}$$

(the origin of y having been shifted) are

$$\eta = A\left(\operatorname{sech}\frac{y}{h}\right)^{\kappa} \cos(kx-\omega t) - \frac{1}{2}\left(\frac{3\kappa}{3\kappa+2}\right) kA^2 \left(\operatorname{sech}\frac{y}{h}\right)^{2\kappa} \tanh\frac{y}{h} \cos 2(kx-\omega t), \tag{6.24}$$

which is equivalent to Thorpe's (1968*b*) equation (3.3.21) if his $f_3 = 0$ and his z is interpreted as above, and

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 + C(\kappa) k^2 A^2, \quad C = \frac{(12\kappa^2+17\kappa+3)\,\Gamma(\kappa+\frac{3}{2})\,\Gamma(2\kappa)}{(3\kappa+2)\,\Gamma(\kappa)\,\Gamma(2K+\frac{5}{2})} \tag{6.25a, b}$$

where

$$\kappa = kh, \quad \omega_1^2 = \left(\frac{\kappa}{\kappa+1}\right)\left(\frac{g'}{2h}\right). \tag{6.26a, b}$$

The coefficient C increases from $\frac{1}{2}$ at $\kappa = 0$ through a rather flat maximum ($0.730 \leqslant C \leqslant 0.736$ for $0.94 \leqslant \kappa \leqslant 3.65$) and then decreases to an asymptote of $2^{-\frac{1}{2}}$ as $\kappa \uparrow \infty$. We remark that the limit $\kappa \downarrow 0$ in (6.24) and (6.25) corresponds to the limit $h_+, h_- \uparrow \infty$ in (5.10) and (5.11).

It should be remarked that the Boussinesq approximation renders the formulation in this section inapplicable to very long, nonlinear (e.g. cnoidal or solitary) waves; cf. Long (1965) and Benjamin (1966). It also implies non-uniform validity in a very deep fluid in which the cumulative density change, albeit gradual, is substantial; cf. Drazin (1969).

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† Dr Thorpe (personal communication) informs me that the substitution $z \rightarrow z_0$ is implicit in his (3.3.4) and (3.3.21).

Appendix. Coupling between surface and internal waves

We now suppose that the upper surface of the two-layer fluid is free, so that (4.1) is replaced by

$$\frac{\mathscr{L}}{S} = \rho_- L_1(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, 0, 0; d_-) + \rho_+ L_1(\boldsymbol{q}_2, \dot{\boldsymbol{q}}_2, \boldsymbol{q}_1, \dot{\boldsymbol{q}}_1; d_+), \tag{A 1}$$

where $\boldsymbol{q}_{1,2} \equiv \{q_n^{1,2}\}$ is the generalized coordinate (matrix) of the interface/free surface. It is expedient to introduce normal modes through the transformation

$$q_n^1 = q_n^{\mathrm{i}} + c_n q_n^{\mathrm{s}}, \quad q_n^2 = q_n^{\mathrm{s}} - \epsilon c_n q_n^{\mathrm{i}}, \tag{A 2a, b}$$

where $q_n^{\mathrm{i,s}}$ are the normal coordinates of the internal/surface waves, ϵ is defined by (4.10) (with $\rho_* \equiv \rho_+$), and the summation convention applies (in this Appendix) only in (A 5). The form of (A 2) ensures the absence of quadratic coupling terms (of the form $q_m^{\mathrm{i}} q_n^{\mathrm{s}}$) in the potential energy. Invoking the corresponding condition (the absence of $\dot{q}_m^{\mathrm{i}} \dot{q}_n^{\mathrm{s}}$) for the kinetic energy, we find that c_n must satisfy

$$\epsilon c_n^2 + \left[\frac{(1-\epsilon)\, a_n^+ + (1+\epsilon)\, a_n^-}{S_n^+ a_n^+}\right] c_n - 1 = 0, \tag{A 3}$$

where $a_n^\pm$ and $S_n^\pm$ are defined by (3.7) for the upper/lower layer, and (as required by the anticipated roles of q_n^{i} and q_n^{s}) that root of (A 3) that is $O(1)$ as $\epsilon \downarrow 0$ (the Boussinesq limit) is to be selected (see below). The quadratic component of the Lagrangian then reduces to

$$\tfrac{1}{2}\delta_{mn}[\ldots] = \tfrac{1}{2} a_n^{\mathrm{s}}[(\dot{q}_n^{\mathrm{s}})^2 - (\omega_n^{\mathrm{s}} q_n^{\mathrm{s}})^2] + \tfrac{1}{2} a_n^{\mathrm{i}}[(\dot{q}_n^{\mathrm{i}})^2 - (\omega_n^{\mathrm{i}} q_n^{\mathrm{i}})^2], \tag{A 4}$$

where (cf. Lamb 1932, §231)

$$a_n^{\mathrm{s}} = (1 - 2c_n S_n^+ + c_n^2)\, a_n^+ + (1+\epsilon)\, c_n^2\, a_n^-, \tag{A 5a}$$

$$a_n^{\mathrm{i}} = (1 + 2\epsilon c_n S_n^+ + \epsilon^2 c_n^2)\, a_n^+ + (1+\epsilon)\, a_n^-, \tag{A 5b}$$

$$(\omega_n^{\mathrm{s}})^2 = (1+\epsilon c_n^2)\frac{g}{a_n^{\mathrm{s}}}, \quad (\omega_n^{\mathrm{i}})^2 = \epsilon(1+\epsilon c_n^2)\frac{g}{a_n^{\mathrm{i}}}. \tag{A 6a, b}$$

Letting $\epsilon \downarrow 0$ in (A 3), (A 5) and (A 6), we obtain

$$c_n \to \frac{S_n^+ a_n^+}{a_n^+ + a_n^-} = \frac{\sinh k_n d_-}{\sinh k_n d}, \quad a_n^{\mathrm{s}} \to \frac{1}{k_n \tanh k_n d}, \quad a_n^{\mathrm{i}} \to a_n^+ + a_n^-, \tag{A 7a, b, c}$$

wherein $d \equiv d_+ + d_-$, and

$$(\omega_n^{\mathrm{s}})^2 \to g k_n \tanh k_n d, \quad (\omega_n^{\mathrm{i}})^2 \to \frac{\epsilon g k_n}{\coth k_n d_+ + \coth k_n d_-}, \tag{A 8a, b}$$

which correspond to surface waves in water of depth d and interfacial waves in an enclosure with a rigid upper boundary (as in §4).

We now address the problem of two-dimensional, nonlinear coupling between a surface wave of wavenumber k and an internal wave of wavenumber $2k$, for which the ψ_n are given by (4.9) and

$$q_n^{\mathrm{s}} = \delta_{1n} q_1, \quad q_n^{\mathrm{i}} = \delta_{2n} q_2. \tag{A 9a, b}$$

Mahony & Smith (1972) have conjectured that quadratic coupling (cubic coupling in the Lagrangian) between these two modes could induce a resonant excitation of the slow mode through heterodyning between the fast mode, externally driven at a frequency $\omega \doteqdot \omega_1^s$, and its sidebands at $\omega \pm \sigma$, where $\sigma \doteqdot \omega_2^i$. (The second harmonic of the surface wave at wavenumber $2k$ will also be quadratically excited, but this component of the full solution has no effect on the internal wave in the present approximation.)

Substituting (3.10a) into (A 1) and invoking (4.9), (A 4) and (A 9), we obtain

$$L = \tfrac{1}{2}a_1^s[\dot{q}_1^2 - (\omega_1^s q_1)^2] + \tfrac{1}{2}a_2^i[\dot{q}_2^2 - (\omega_2^i q_2)^2] + a_{112}\, q_1 \dot{q}_1 \dot{q}_2 + \tfrac{1}{2}a_{211}\, q_2 \dot{q}_1^2 + L_4, \quad \text{(A 10)}$$

where the coefficients of the quadratic terms are given by (A 5) and (A 6),

$$a_{112} = a_{121} = \epsilon C_{112}(c_1^2 - c_2) + D_{112}\{a_1^+ a_2^+[(1 - c_1 S_1^+)(\epsilon c_2 + S_2^+) + c_1(c_1 - S_1^+)(1 + \epsilon c_2 S_2^+)] - (1+\epsilon)\, a_1^- a_2^- c_1^2\}, \quad (A\ 11a)$$

$$a_{211} = \epsilon C_{112}(c_1^2 - c_2) + D_{211}\{a_1^{+2}[(c_1 - S_1^+)^2 + \epsilon c_2(1 - c_1 S_1^+)^2] - (1+\epsilon)\, a_1^{-2} c_1^2\}, \quad (A\ 11b)$$

and the quartic component L_4 is implicitly determined by (3.9b), (3.10b), and (A 9a, b). It follows from (A 11) (after considerable algebraic reduction) that a_{112} and a_{211} are $O(\epsilon)$ and therefore negligible in the Boussinesq approximation, in which $O(\epsilon)$ is retained only in ω_n^i (A 8b). It also can be shown that a_{112} and a_{211} vanish in the limit of a deep lower layer ($kd_- \uparrow \infty$). It therefore appears that the putative resonance between the surface and internal waves is realized only at higher order if at all.†

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† It is the fact that both $(\omega_2^i/\omega_2^s)^2$ and the quadratic coupling are $O(\epsilon)$, rather than simply the failure of the Boussinesq approximation, that negates the Mahony–Smith resonance.

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